Critical behavior and limit theorems for self-avoiding walk in high dimensions: The lace-expansion approach

Akira Sakai*

November 11-13, 2008

Abstract

Self-avoiding walk (SAW) is a statistical-mechanical model for a linear polymer in a good solvent. We consider SAW on $\mathbb{Z}^d$ and say that a path $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ of length $|\omega| = n$ is self-avoiding if $\omega_i \neq \omega_j$ for all $i \neq j$. Let

$$c_n(x) = \sum_{\substack{\omega: o \to x \\ |\omega|=n}} 1_{\{\omega \text{ is self-avoiding}\}} \prod_{i=1}^{n} D(\omega_i - \omega_{i-1}), \qquad c_0(x) = \delta_{o,x},$$

where D is a $\mathbb{Z}^d$-symmetric probability distribution. If the indicator $1_{\{\omega \text{ is self-avoiding}\}}$ is absent, then $c_n(x)$ is reduced to the n-step transition probability for random walk generated by the 1-step distribution D. Let $\hat{c}_n(k) = \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x} c_n(x)$.

Many researchers in mathematics and physics have been attracted to this model, due to the fact that it exhibits critical behavior. Since its introduction, SAW has been investigated actively for decades. For the nearest-neighbor model (i.e., $D(x) \propto 1_{\{|x|=1\}}$), in particular, Hara and Slade finally prove in 1992 using the *lace expansion* that it exhibits Gaussian mean-field behavior for all $d \geq 5$: e.g., $\exists A = A(d) \in (0, \infty)$ such that $\hat{c}_n(k/\sqrt{n})/\hat{c}_n(0) \to e^{-A|k|^2}$ as $n \to \infty$. For $d \leq 4$, on the other hand, we do not expect such simple behavior. In fact, in 2 dimensions, it has been discovered that, if there is a conformal-invariant continuum-limit, it must be described by the Schramm-Loewner evelution with parameter 8/3 ($\mathrm{SLE}_{8/3}$ for short). At the upper-critical dimension $d = 4$, it has been conjectured that there are log corrections to the Gaussian mean-field behavior. Brydges and Slade have been working on this issue using a renormalization-group method. (A seminar series by Slade is planned in winter 2009 at Kyoto University.)

In my series of talks, I will explain the most recent and most efficient form of the lace-expansion analysis, which is also applicable to long-range SAW defined by $D(x) \approx |x/L|^{-d-\alpha}$ with $\alpha > 0$, where $L < \infty$ is the spread-out parameter that is taken to be large in the analysis. Notice that the variance of D does not exist if $\alpha \leq 2$. Among the results obtained so far is the following limit theorem: $\forall d > 2(\alpha \wedge 2)$ and $L \gg 1$ (depending on d and α), $\exists A = A(d, \alpha, L) \in (0, \infty)$ such that

$$\lim_{n\to\infty} \frac{\hat{c}_n(k_n)}{\hat{c}_n(0)} = e^{-A|k|^{\alpha\wedge 2}}, \qquad \text{where} \quad k_n = k \times \begin{cases} n^{-\frac{1}{\alpha \wedge 2}} & (\alpha \neq 2), \\ (n \log n)^{-1/2} & (\alpha = 2). \end{cases}$$

The material covered is based on my recent joint work with Chen, Heydenreich, and van der Hofstad.

*Creative Research Initiative "Sousei", Hokkaido University, North 21, West 10, Kita-ku, Sapporo 001-0021, Japan. http://www.math.sci.hokudai.ac.jp/∼sakai/